A novel view on the physical origin of E_8

2008 J. Phys. A: Math. Theor. 41 332001

(http://iopscience.iop.org/1751-8121/41/33/332001)

IOP FTC ▶▶▶
IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 332001 (10pp) doi:10.1088/1751-8113/41/33/332001

FAST TRACK COMMUNICATION

A novel view on the physical origin of E_8

Matej Pavšič

Jožef Stefan Institute, Jamova 39, 1000 Ljubljana, Slovenia

E-mail: matej.pavsic@ijs.si

Received 20 May 2008, in final form 26 June 2008
Published 11 July 2008
Online at stacks.iop.org/JPhysA/41/332001

Abstract
We consider a straightforward extension of the four-dimensional spacetime M_4 to the space of extended events associated with strings/branes, corresponding to points, lines, areas, 3-volumes and 4-volumes in M_4. All those objects can be elegantly represented by the Clifford numbers $X \equiv x^A\gamma_A \equiv x^{a_1\ldots a_r}\gamma_{a_1\ldots a_r}$, $r = 0, 1, 2, 3, 4$. This leads to the concept of the so-called Clifford space $\mathcal{C}$, a 16-dimensional manifold whose tangent space at every point is the Clifford algebra $\mathcal{C}\ell(1, 3)$. The latter space besides an algebra is also a vector space whose elements can be rotated into each other in two ways: (i) either by the action of the rotation matrices of $SO(8, 8)$ on the components x^A or (ii) by the left and right action of the Clifford numbers $R = \exp[\alpha^A\gamma_A]$ and $S = \exp[\beta^A\gamma_A]$ on X. In the latter case, one does not recover all possible rotations of the group $SO(8, 8)$. This discrepancy between the transformations (i) and (ii) suggests that one should replace the tangent space $\mathcal{C}\ell(1, 3)$ with a vector space $V_{8,8}$ whose basis elements are generators of the Clifford algebra $\mathcal{C}\ell(8, 8)$, which contains the Lie algebra of the exceptional group E_8 as a subspace. E_8 thus arises from the fact that, just as in the spacetime M_4 there are r-volumes generated by the tangent vectors of the spacetime, there are R-volumes, $R = 0, 1, 2, 3, \ldots, 16$, in the Clifford space $\mathcal{C}$, generated by the tangent vectors of $\mathcal{C}$.

PACS numbers: 02.20.−a, 04.50.−h

1. Introduction

The unification of fundamental interactions has turned out to be an elusive goal. Amongst the numerous approaches that have been pursued, Kaluza–Klein and string/brane theories are of particular importance. While Kaluza–Klein theories [1] introduce extra dimensions into the game, string/brane theories [2] incorporate the extended nature of physical objects. Although very promising, these theories have encountered serious problems that have remained unresolved, and so far, they have not led to a realistic model. In order to develop such a model, one has to find a grand unification group, e.g., $SU(5)$, $SO(10)$, E_6 or E_8, which fits into a more fundamental theory that also incorporates the Poincaré group of spacetime. For the time being, a lot of research is devoted to explaining the properties of these groups regardless of their possible deeper origin in physics.

Last year, Lisi [3] proposed that one could unify all the fields of the Standard model and gravity by means of the exceptional group E_8. Lisi's approach differs from others [4] in attempting to explain generations in terms of triality. It has attracted considerable attention and criticism. Regardless of whether the particular model due to Lisi will turn out to be correct or not, it would be desirable to find a possible physical basis for the group E_8. In other words, we would like to know whether there exists a particular physical arrangement that leads to E_8. Is it due to the existence of higher-dimensional dynamics? We will show that we do not need extra dimensions of spacetime in order to arrive at E_8. We can remain in the four-dimensional spacetime M_4 and consider not only events (points) but also oriented lines, areas, 3-volumes and 4-volumes in M_4. All these objects can elegantly be represented by the Clifford numbers $X^A\gamma_A \equiv X^{a_1a_2\ldots a_r}\gamma_{a_1a_2\ldots a_r}$, where $r = 0, 1, 2, 3, 4$ and $a = 0, 1, 2, 3$. This leads to the concept of the so-called *Clifford space* [5–9], a 16-dimensional manifold whose tangent space is the Clifford algebra[1] $C\ell(4)$.

In this paper, it has been assumed that the arena for physics is Clifford space $\mathcal{C}$ and that physical objects are described by Clifford algebra valued fields on $\mathcal{C}$. At every point $X \in \mathcal{C}$, a field Φ can be transformed according to [10, 11]

$$\Phi \to \Phi' = \mathrm{R}\Phi\mathrm{S}, \tag{1}$$

where R and S are in general two different Clifford numbers [10, 11] to be specified later. For the time being, let us only say that the transformation (1), in general, reshuffles the grade r of the components $\Phi = \phi^A\gamma_A$. Its action on ϕ^A or γ_A can just be a *rotation*. Altogether (if $D = 2^n$ is the dimension of $C\ell(n)$) there are $D(D-1)/2$ independent rotations, 120 if $n = 4$, but they cannot all be generated by the 16 independent γ_A which close $C\ell(4)$. However, we can perform rotations in $C\ell(4)$ by means of the rotation matrices acting on ϕ^A according to

$$\phi'^A = L^A{}_B\phi^B. \tag{2}$$

The existence of 120 rotational degrees of freedom implies the presence of extra degrees of freedom besides those spanned by the basis $\{\gamma_A\}$.

At this point, we will employ the fact that the Clifford algebra $C\ell(4)$ is a vector space. Equation (1) performs only 32 independent rotations in the vector space (16 due to the left and the remaining 16 due to the right transformations). Therefore, instead of the basis $\{\gamma_A\}$, we may use an alternative basis $\{\Gamma_A\}$, $A = 1, 2, \ldots, 16$, whose elements Γ_A generate the Clifford algebra $C\ell(16)$. The new basis elements Γ_A span a 16-dimensional space V_{16} which, as a vector space, is isomorphic to $C\ell(4)$. Rotations in V_{16} are generated by bivectors $\Gamma_{AB} \equiv C\ell(16)$; they act on a vector $\phi^A\Gamma_A \in V_{16}$ and belong to the Lie algebra of the group $SO(16)$. The latter group has not only a vector representation, but also spinor representations. In our set up spinors come from the Clifford algebra $C\ell(16)$, namely, from its left (right) minimal ideals. The Lie algebra so(16), together with the space S^+_{16} of real, positive chiral $C\ell(16)$ spinors, forms the Lie algebra e_8 of E_8.

So, we have found that e_8 arises from the spacetime M_4 itself, from which one can construct a Clifford algebra. The latter algebra, as a vector space, is isomorphic to V_{16}, and its basis vectors Γ_A can generate the Clifford algebra $C\ell(16)$, whose subspace is e_8:

$$V_4 \to C\ell(4) \sim V_{16} \to C\ell(16) \supset e_8. \tag{3}$$

In the following, we will explore this procedure in more detail.

[1] In the introduction, we used a simplified notation for various spaces, namely M_4, V_4, $C\ell(4)$, V_{16} and $C\ell(16)$, even though the Minkowski space M_4 has signature (1,3) and should be written more precisely as $M_{1,3}$, V_4 as $V_{1,3}$, $C\ell(4)$ as $C\ell(1,3)$, V_{16} as $V_{8,8}$ and $C\ell(16)$ as $C\ell(8,8)$.

2. Generalizing spacetime

Special and general relativity are valid in the spacetime manifold M_4. Choosing a point $\mathcal{P} \in M_4$, the tangent space at that point is a vector space $V_4 \equiv T_{\mathcal{P}}(M_4)$, whose basis vectors can be taken to be generators γ_a, $a = 0, 1, 2, 3$, of the Clifford algebra, satisfying

$$\gamma_a \cdot \gamma_b \equiv \tfrac{1}{2}(\gamma_a\gamma_b + \gamma_b\gamma_a) = \eta_{ab}, \tag{4}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric tensor.

While the symmetrized product (4) gives the metric, the antisymmetrized (wedge) product,

$$\gamma_a \wedge \gamma_b \equiv \tfrac{1}{2}[\gamma_a, \gamma_b] = \tfrac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a), \tag{5}$$

gives a bivector. In general, the antisymmetrized product of r vectors

$$\gamma_{a_1a_2\ldots a_r} \equiv \gamma_{a_1} \wedge \gamma_{a_2} \wedge \cdots \wedge \gamma_{a_r} \equiv \frac{1}{r!}[\gamma_{a_1}, \gamma_{a_2}, \ldots, \gamma_{a_r}] \equiv \gamma_A \tag{6}$$

gives an r-vector. The objects $\gamma_{a_1a_2\ldots a_r} \equiv \gamma_A$ form a basis of the Clifford algebra $\mathcal{C}\ell(1, 3)$, whose elements are Clifford numbers called *polyvectors* and which are given by a superposition

$$X = X^{a_1\ldots a_r}\gamma_{a_1\ldots a_r} \equiv X^A\gamma_A. \tag{7}$$

We take $a_1 < a_2 < \cdots < a_r$ in order to avoid multiple occurrences of the same term.

The components $X^{a_1\ldots a_r} \equiv X^A$ are holographic projections of oriented r-areas onto the coordinate r-planes, spanned by $\gamma_{a_1\ldots a_r} \equiv \gamma_A$.

In the spacetime manifold M_4, in which we choose a point $\mathcal{P}_0$ as the origin, the vectors $x = x^a\gamma_a \in V_4(\mathcal{P}_0) = T_{\mathcal{P}_0}(M_4)$ are in one-to-one correspondence with other points $\mathcal{P} \in M_4$, at least within a region $B \subset M_4$ around $\mathcal{P}_0$.

Similarly, the polyvectors $X = X^A\gamma_A \in \mathcal{C}\ell(1, 3)$ are in one-to-one correspondence with the points $\mathcal{E}$ of a 16-dimensional manifold $\mathcal{C}$, called a *Clifford space*, in which we choose a point $\mathcal{E}_0$ as the origin, such that $\gamma_A \in T_{\mathcal{E}_0}(\mathcal{C})$ are tangent polyvectors at $\mathcal{E}_0$. In other words, vectors of the tangent space at $\mathcal{E}_0 \in \mathcal{C}$ can be mapped into the points $\mathcal{E} \in \mathcal{C}$ of a region around $\mathcal{E}_0$. A tangent space to a point $\mathcal{E} \in \mathcal{C}$ is the vector space defined by the Clifford algebra $\mathcal{C}\ell(1, 3)$.

The metric is given by the scalar product

$$G_{AB} = \gamma_A^\ddagger * \gamma_B \equiv \langle\gamma_A^\ddagger\gamma_B\rangle_0. \tag{8}$$

Here $\ddagger$ reverses the order of the 1-vectors—for example, $\left(\gamma_{a_1}\gamma_{a_2}\ldots\gamma_{a_r}\right)^\ddagger = \gamma_{a_r}\ldots\gamma_{a_2}\gamma_{a_1}$—and is necessary for the consistent raising and lowering of indices [11, 15]. With the definition (8), the signature of $\mathcal{C}$ is (8, 8).

Since a Clifford space is such a straightforward generalization of spacetime, its has been assumed [5–9] that physics also has to be generalized. According to such a novel theory, the arena for physics is not the spacetime M_4 but the Clifford space $\mathcal{C}$ (also called $\mathcal{C}$-space). The fundamental building blocks of physical space are then not the points of spacetime but extended r-dimensional cells (r-cells), or extended events described by r-vectors. For definiteness, we may imagine that extended events are associated with p-branes in the quenched minisuperspace description [16, 17], but in principle, extended events can be related to whichever fundamental extended objects that can be described by r-vectors.

3. Group $SO(8, 8)$ acting in Clifford space

3.1. Left and right transformations of a polyvector

A polyvector $X \in \mathcal{C}\ell(1, 3)$ can be transformed into another polyvector $X' \in \mathcal{C}\ell(1, 3)$ by

$$X' = \mathrm{R}X\mathrm{S}, \tag{9}$$

where R, S $\in \mathcal{C}\ell(1,3)$. Requiring that the norm of X be invariant

$$|X'|^2 \equiv X'^{\ddagger} * X' \equiv \langle X'^{\ddagger} X' \rangle_0 = \langle \mathrm{S}^{\ddagger} X^{\ddagger} \mathrm{R}^{\ddagger} \mathrm{R} X \mathrm{S} \rangle_0 = \langle X^{\ddagger} X \rangle_0, \tag{10}$$

we have

$$\mathrm{R}^{\ddagger}\mathrm{R} = 1, \qquad \mathrm{S}^{\ddagger}\mathrm{S} = 1. \tag{11}$$

Writing

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha^A \gamma_A}, \qquad \mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta^A \gamma_A}, \tag{12}$$

we have from equation (11) that

$$\alpha^* \gamma_A^{\ddagger} = -\gamma_A \alpha^A, \tag{13}$$

where we assume that reversion also acts on the imaginary unit i, which we consider a bivector $\mathrm{i} = e_q e_p$ of the phase space spanned by e_q and e_p [8]. The parameters $\alpha^A, \beta^A, A = 1, 2, \ldots, 16$ are, therefore, either real or imaginary because, depending on the grade of γ_A, we have

$$\gamma_A^{\ddagger} = \gamma_A \quad \text{if} \quad A \in \{A\}_+, \qquad \gamma_A^{\ddagger} = -\gamma_A \quad \text{if} \quad A \in \{A\}_-. \tag{14}$$

As an example, let us consider the case

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\alpha\gamma_1\gamma_2} = \cos\frac{\alpha}{2} + \gamma_1\gamma_2 \sin\frac{\alpha}{2}, \qquad \mathrm{S} = \mathrm{e}^{\frac{1}{2}\beta\gamma_1\gamma_2} = \cos\frac{\beta}{2} + \gamma_1\gamma_2 \sin\frac{\beta}{2} \tag{15}$$

and examine how various Clifford numbers $X = X^C \gamma_C$ transform under (9).

(i) If $X = X^1\gamma_1 + X^2\gamma_2$, then the transformation (9) gives

$$X' = X^1\left(\gamma_1 \cos\frac{\alpha-\beta}{2} + \gamma_2 \sin\frac{\alpha-\beta}{2}\right) + X^2\left(-\gamma_1 \sin\frac{\alpha-\beta}{2} + \gamma_2 \cos\frac{\alpha-\beta}{2}\right). \tag{16}$$

(ii) If $X = X^3\gamma_3 + X^{123}\gamma_{123}$, then

$$X' = X^3\left(\gamma_3 \cos\frac{\alpha+\beta}{2} + \gamma_{123} \sin\frac{\alpha+\beta}{2}\right) + X^{123}\left(-\gamma_3 \sin\frac{\alpha+\beta}{2} + \gamma_{123} \cos\frac{\alpha+\beta}{2}\right). \tag{17}$$

(iii) If $X = s\underline{1} + X^{12}\gamma_{12}$, then

$$X' = s\left(\underline{1} \cos\frac{\alpha+\beta}{2} + \gamma_{12} \sin\frac{\alpha+\beta}{2}\right) + X^2\left(-\underline{1} \sin\frac{\alpha+\beta}{2} + \gamma_2 \cos\frac{\alpha+\beta}{2}\right). \tag{18}$$

(iv) If $X = \tilde{X}^1\gamma_5\gamma_1 + \tilde{X}^2\gamma_5\gamma_2$, then

$$X' = \tilde{X}^1\left(\gamma_5\gamma_1 \cos\frac{\alpha-\beta}{2} + \gamma_5\gamma_2 \sin\frac{\alpha-\beta}{2}\right) + \tilde{X}^2\left(-\gamma_5\gamma_1 \sin\frac{\alpha-\beta}{2} + \gamma_5\gamma_2 \cos\frac{\alpha-\beta}{2}\right). \tag{19}$$

The usual rotations of vectors or pseudovectors are reproduced if the angle β for the right transformation is equal to minus the angle α for the left transformation, i.e., if

$$\beta = -\alpha, \tag{20}$$

then all other transformations which mix the grade vanish.

Condition (20) is a prerequisite for the consistency of ordinary rotations and Lorentz transformations. If we release the condition (20) and allow for arbitrary independent α and β, in particular $\alpha = \beta$, then we go outside the Lorentz group in M_4 and arrive at rotations in Clifford space.

In the example (15), the generator $\gamma_1\gamma_2$ changes sign under reversion, so the parameters α and β are real. If we consider, for example, the rotations generated by γ_1, then, since $\gamma_1^\ddagger = \gamma_1$, the transformation parameter must be imaginary, which we write explicitly as

$$\mathrm{R} = \mathrm{e}^{\frac{1}{2}\mathrm{i}\alpha\gamma_1} = \mathrm{ch}\,\frac{\alpha}{2} + \mathrm{i}\gamma_1\,\mathrm{sh}\,\frac{\alpha}{2}, \qquad \mathrm{S} = \mathrm{e}^{\frac{1}{2}\mathrm{i}\beta\gamma_1} = \mathrm{ch}\,\frac{\beta}{2} + \mathrm{i}\gamma_1\,\mathrm{sh}\,\frac{\beta}{2}. \tag{21}$$

Taking $X = s\underline{1}$ or $X = X^1\gamma_1$, we have

$$\mathrm{R}\,s\underline{1}\mathrm{S} = s\left(\underline{1}\,\mathrm{ch}\,\frac{\alpha+\beta}{2} + \mathrm{i}\gamma_1\,\mathrm{sh}\,\frac{\alpha+\beta}{2}\right) \tag{22}$$

$$\mathrm{R}X^1\gamma_1\underline{1}\mathrm{S} = X^1\left(-\mathrm{i}\underline{1}\,\mathrm{ch}\,\frac{\alpha+\beta}{2} + \gamma_1\,\mathrm{sh}\,\frac{\alpha+\beta}{2}\right). \tag{23}$$

We see that $\mathrm{i}\gamma_1$ generates rotations in the plane $(\mathrm{i}\underline{1}, \gamma_1)$ or $(\underline{1}, \mathrm{i}\gamma_1)$.

For infinitesimal transformations we have

$$X' = \mathrm{e}^{\frac{1}{2}\alpha^A\gamma_A}X^C\gamma_C\,\mathrm{e}^{\frac{1}{2}\beta^B\gamma_B} \approx 1 + \delta X, \tag{24}$$

where

$$\begin{aligned}\delta X = \delta X^C\gamma_C &= \frac{1}{2}(\alpha^A\gamma_A\gamma_C + \beta^A\gamma_C\gamma_A)X^C\\ &= \frac{\alpha^A+\beta^A}{4}\{\gamma_A,\gamma_C\}X^C + \frac{\alpha^A-\beta^A}{4}[\gamma_A,\gamma_C]X^C\\ &= \frac{\alpha^A+\beta^A}{4}f_{AC}^D\gamma_D X^C + \frac{\alpha^A-\beta^A}{4}C_{AC}^D\gamma_D X^C,\end{aligned} \tag{25}$$

where $\{\gamma_A,\gamma_C\} = f_{AC}^D\gamma_D$ and $[\gamma_A,\gamma_C] = C_{AC}^D\gamma_D$. Introducing

$$\epsilon^D{}_C = \frac{\alpha^A+\beta^A}{4}f_{AC}^D\gamma_D + \frac{\alpha^A-\beta^A}{4}C_{AC}^D\gamma_D, \tag{26}$$

we find

$$\delta X^D = \epsilon^D{}_C X^C, \tag{27}$$

which denotes infinitesimal rotations of the $\mathcal{C}$-space coordinates X^C if $\epsilon^D{}_C$ is antisymmetric, and other linear transformations otherwise. Equation (26) implies that there are only 32 independent components of $\epsilon^D{}_C$, of which 16 are due to α^A, and 16 to β^A.

A single generator, e.g., γ_{12} in examples (15)–(19), generates rotations not in a single plane but in several planes of $\mathcal{C}$-space at once. An additional complication is that, in order to satisfy the requirement (11) for the invariance of the norm, some transformation parameters must be real, and some must be imaginary. Therefore, some rotations are within $\mathcal{C}\ell(1,3)$ while others are between $\mathcal{C}\ell(1,3)$ and $\mathrm{i}\mathcal{C}\ell(1,3)$, as illustrated in (21)–(23). Our space is then enlarged to $\mathbb{C}\otimes\mathcal{C}\ell(1,3)$.

So, we have extended real polyvectors to complex ones. In quantized theories, physical particles are described by fields that, in general, are complex valued. In our case, these fields are $\Phi = \phi^A\gamma_A$, whose values are in a complexified Clifford algebra. The quadratic form is

$$\Phi^\ddagger * \Phi = \phi^{\ddagger A}\phi^B G_{AB} = \phi_{(R)}^A\phi_{(R)}^B G_{AB} + \phi_{(I)}^A\phi_{(I)}^B G_{AB}, \tag{28}$$

where G_{AB} is defined according to equation (8).

The real components $\phi_{(R)}^A$ can be transformed amongst themselves by rotations of the group $SO(8,8)$, and the same is true for the imaginary components:

$$\phi_{(R)}'^A = L^A{}_B\phi_{(R)}^B, \qquad \phi_{(I)}'^A = L^A{}_B\phi_{(I)}^B. \tag{29}$$

There are also mixed transformations between $\phi^A_{(R)}$ and $\phi^A_{(I)}$. The Lie algebra of $SO(8,8)$ has 120 independent generators. Altogether, the group of transformations that preserve the quadratic form (28) has 120 dimensions due to rotations of $\phi^A_{(R)}$, 120 dimensions due to rotations of $\phi^A_{(I)}$ and $16 \times 16 = 256$ dimensions belonging to mixed transformations between $\phi^A_{(R)}$ and $\phi^A_{(I)}$, which sums to 496. This, of course, is in agreement with the dimension $32 \times 31/2 = 496$ of the $SO(16,16)$ acting on $\left(\phi^A_{(R)}, \phi^A_{(I)}\right)$.

3.2. Vector space $V_{8,8}$

We have seen that, amongst left and right transformations of a real polyvector X, there are norm-preserving transformations, i.e., rotations that preserve or change the grade of the elements $\gamma_A \in \mathcal{C}(1,3)$. Therefore, besides the rotations in the planes (γ_μ, γ_ν) and $(\gamma_5\gamma_\mu, \gamma_5\gamma_\nu)$, we also have rotations in the planes $(\gamma_\mu, \gamma_{\rho\sigma})$ spanned by vectors and bivectors, etc. Such grade-mixing rotations generated by (9) do not act in a single plane but in several planes at once, as illustrated in examples (15)–(19). Moreover, there are planes in $\mathcal{C}\ell(1,3)$ in which rotations cannot be generated by a transformation (9). For instance, a rotation in the plane $(\underline{1}, \gamma_1)$ does not occur in (9). Instead, one has a rotation in the plane $(\underline{1}, \mathrm{i}\gamma_1)$, which is not in $\mathcal{C}\ell(1,3)$, but in $\mathbb{C} \otimes \mathcal{C}\ell(1,3)$.

A basis $\gamma_A \in \mathcal{C}\ell(1,3)$ cannot generate all possible rotations of the group $SO(8,8)$ acting within $\mathcal{C}\ell(1,3)$. However, we can construct the $SO(8,8)$ rotations directly, without using γ_A as generators:

$$X' = X'^A\gamma_A = L^A{}_B X^B\gamma_A = X^A\gamma'_A, \tag{30}$$

where

$$X'^A = L^A{}_B X^B, \qquad \gamma'_A = L^B{}_A\gamma_B. \tag{31}$$

So, either the components X^A or the basis (poly)vectors γ_A can be rotated by a matrix $L^A{}_B$ of $SO(8,8)$.

If we consider $\mathcal{C}\ell(1,3)$ as a vector space $V_{8,8}$ and forget about the algebra, then, instead of $\gamma_A \in \mathcal{C}\ell(1,3)$, we can use any other elements that span $V_{8,8}$. Let us choose as our basis vectors the generators Γ_A, $A = 1, 2, \ldots, 16$ of $\mathcal{C}\ell(8,8)$. The generators of rotations in $V_{8,8}$ are then the 120 independent bivectors $\Gamma_{AB} = \Gamma_A \wedge \Gamma_B \in \mathrm{so}(8,8) \subset \mathcal{C}\ell(8,8)$.

Our space is thus the 16-dimensional vector space $V_{8,8}$, spanned by the vectors $\Gamma_A \in \mathcal{C}\ell(8,8)$. An element $X = X^A\Gamma_A \in V_{8,8}$ is rotated according to

$$X' = \mathrm{e}^{\frac{1}{4}\alpha^{AB}\Gamma_{AB}} X\, \mathrm{e}^{-\frac{1}{4}\alpha^{AB}\Gamma_{AB}}. \tag{32}$$

By also taking into account the $\mathcal{C}\ell(8,8)$ spinors, we arrive at the Lie algebra e_8 as a subspace of $\mathcal{C}\ell(8,8)$.

4. Vector space $V_8^{(1)} \oplus V_8^{(2)}$

Now, we will describe an alternative procedure that also leads from $\mathcal{C}\ell(1,3)$ to e_8. We can split $\mathcal{C}\ell(1,3) \sim V_{8,8}$ into two subspaces $V_{2,6}$ and $V_{6,2}$ spanned by the following basis elements:

$$\begin{aligned} &V_{2,6}: \quad \{\gamma_{A_1}\} = \{1, \gamma_\mu, \gamma_{0r}\}, \qquad && A_1 = 1, 2, \ldots, 8 \\ &V_{6,2}: \quad \{\gamma_{A_2}\} = \{\gamma_5, \gamma_5\gamma_\mu, \gamma_5\gamma_{0r}\}, \qquad && A_2 = 9, 10, \ldots, 16, \end{aligned} \tag{33}$$

where the three elements $\gamma_5\gamma_{0r}$, $r = 1, 2, 3$, correspond to the three elements γ_{rs}. Every element $\Phi \in \mathcal{C}\ell(1,3)$ can thus be split into two parts in the following way:

$$\Phi = \phi^A\gamma_A = \phi^{A_1}\gamma_{A_1} + \phi^{A_2}\gamma_{A_2}. \tag{34}$$

For convenience, we perform the split $\mathcal{C}\ell(1,3) \sim V_{8,8} = V_{2,6} \oplus V_{6,2}$. Alternative splits, such as $V_{8,8} = V_{0,8} \oplus V_{8,0} = V_{1,7} \oplus V_{7,1}$, etc, are also possible in principle. For simplicity, we will from now onwards use the symbol V_8 for all these possible spaces, and consider the generic split

$$\mathcal{C}\ell(1,3) \sim V_{8,8} = V_8^{(1)} \oplus V_8^{(2)}. \tag{35}$$

The basis elements $\gamma_A \equiv \gamma_{a1\ldots a_r}$, $r = 0, 1, 2, 3, 4$, given by equation (6), are r-vectors of $\mathcal{C}\ell(1,3)$. Instead of a basis consisting of 16 r-vectors, we can consider a basis that consists of 16 independent spinors $\xi_{\tilde{A}} \equiv \xi_{\alpha i}$, $\alpha = 1, 2, 3, 4$, $i = 1, 2, 3, 4$, that span four independent left minimal ideals of $\mathcal{C}\ell(1,3)$ [11, 18]. Then, Φ can be expanded according to

$$\Phi = \psi^{\tilde{A}}\xi_{\tilde{A}} = \psi^{\alpha 1}\xi_{\alpha 1} + \psi^{\alpha 2}\xi_{\alpha 2} + \psi^{\alpha 3}\xi_{\alpha 3} + \psi^{\alpha 4}\xi_{\alpha 4}. \tag{36}$$

Four linearly independent spinors of one minimal left ideal form a four-dimensional representation of $\mathcal{C}\ell(1,3)$. In terms of $\xi_{\alpha i}$, for a fixed i, say $i = 1$, every element of $\mathcal{C}\ell(1,3)$ can be represented as a 4×4 matrix. In terms of all 16 basis elements $\xi_{\tilde{A}}$, every element $\Phi \in \mathcal{C}\ell(1,3)$ can be represented as a 16×16 matrix according to [11]:

$$\langle \xi^{\tilde{A}\ddagger}\Phi\xi_{\tilde{B}}\rangle_S = (\Phi)^{\tilde{A}}{}_{\tilde{B}} = \delta^i{}_j \otimes (\Phi)^\alpha{}_\beta, \tag{37}$$

where α, β are the 4-spinor indices of a given left minimal ideal and the bracket with subscript S denotes the normalized scalar part [11]. The above formula explicitly shows that the 16-dimensional representation of $\mathcal{C}\ell(1,3)$ is reducible: the matrices are block diagonal, with the blocks being just 4×4 matrices. In general, instead of 16 spinors $\xi_{\tilde{A}} \equiv \xi_{\alpha i}$, one can use some other set of 16 linearly independent objects q_A, each of which is a superposition of $\xi_{\tilde{A}}$. A matrix

$$(\Phi)^A{}_B = \langle q^{A\ddagger}\Phi q_B\rangle_S, \tag{38}$$

then may not be block diagonal, but, of course, can be transformed back into block-diagonal form. If $q_A = \gamma_A$, then $\langle \gamma^{A\ddagger}\Phi\gamma_B\rangle_S$ is a representation of a Clifford number Φ in the Clifford algebra basis γ_A.

Let us now generalize from *reducible* 16×16 matrices to *arbitrary* 16×16 matrices. An arbitrary 16×16 matrix can be expanded in terms of 256 linearly independent 16×16 matrices that represent a basis of the Clifford algebra $\mathcal{C}\ell(8)$ (or one of its non-compact forms), where $256 = 2^8$ is the dimension of $\mathcal{C}\ell(8)$ and $16 = 2^{8/2}$ is the dimension of a corresponding space that forms an irreducible representation of $\mathcal{C}\ell(8)$.

Let us now return to the split (35). Instead of the basis elements $\gamma_{A_1}, \gamma_{A_2} \in \mathcal{C}\ell(1,3)$ that, according to (37) and (38), can be represented by reducible (block diagonal) 16×16 matrices, let us now consider new basis elements $\Gamma^{(1)}_{\bar{A}}$ and $\Gamma^{(2)}_{\bar{A}}$, $\bar{A} = 1, 2, \ldots, 8$, that are 1-vectors of $\mathcal{C}\ell(8)$, represented by 16×16 matrices that, in general, are not reducible to a block-diagonal form (37). Every element of $V_8^{(1)}$ can now be expanded in terms of $\Gamma^{(1)}_{\bar{A}}$, and analogously for $V_8^{(2)}$ and $\Gamma^{(2)}_{\bar{A}}$.

Besides the 1-vectors $\Gamma^{(1)}_{\bar{A}} \in V_8^{(1)} \subset \mathcal{C}\ell\big(V_8^{(1)}\big)$ and $\Gamma^{(2)}_{\bar{A}} \in V_8^{(2)} \subset \mathcal{C}\ell\big(V_8^{(2)}\big)$, which are the Clifford algebra generators of the two copies of $\mathcal{C}\ell(8)$, let us consider also 2-vectors

$$\Gamma^{(1)}_{\bar{A}\bar{B}} \in \mathrm{so}\big(V_8^{(1)}\big) \qquad \text{and} \qquad \Gamma^{(2)}_{\bar{A}\bar{B}} \in \mathrm{so}\big(V_8^{(2)}\big).$$

The latter objects are generators of rotation in $V_8^{(1)}$ and $V_8^{(2)}$, respectively.

In addition to the rotations in each of the spaces $V_8^{(1)}$, $V_8^{(2)}$, there are also transformations that mix those two spaces, and they are given by the generators $\Gamma^{(1)}_{\bar{A}} \otimes \Gamma^{(2)}_{\bar{A}}$. Altogether, there are 28 generators of $\mathrm{so}\big(V_8^{(1)}\big)$, 28 generators of $\mathrm{so}\big(V_8^{(2)}\big)$, and $8 \times 8 = 64$ generators

$\Gamma_{\bar{A}}^{(1)} \otimes \Gamma_{\bar{A}}^{(2)}$, which sums to a total of 120 generators. This is the dimension of so(16) that can be decomposed according to [13, 14]:

$$\mathrm{so}(16) = \mathrm{so}\big(V_8^{(1)}\big) \oplus \mathrm{so}\big(V_8^{(2)}\big) \oplus V_8^{(1)} \otimes V_8^{(2)}. \tag{39}$$

In addition to 1-vectors and 2-vectors in $\mathcal{C}\ell(8)$, we also have spinors as elements of left (right) minimal ideals of $\mathcal{C}\ell(8)$. The dimension of a given minimal left ideal of $\mathcal{C}\ell(8)$ is $2^{8/2} = 16$. There are 16 linearly independent basis spinors ζ_A, $A = 1, 2, \ldots, 16$, and they span the spinor space S_8 that decomposes into two invariant subspaces S_8^+ and S_8^- forming, respectively, the left-handed and right-handed spinor representation of so(8). From the basis spinors of $\mathcal{C}\ell\big(V_8^{(1)}\big)$ and $\mathcal{C}\ell\big(V_8^{(2)}\big)$, we can form the following objects that live in $\mathcal{C}\ell(16) = \mathcal{C}\ell(8) \otimes \mathcal{C}\ell(8)$:

$$\zeta_{\bar{A}}^{+(1)} \otimes \zeta_{\bar{B}}^{+(2)}, \qquad \zeta_{\bar{A}}^{-(1)} \otimes \zeta_{\bar{B}}^{-(2)}, \tag{40}$$

$$\zeta_{\bar{A}}^{-(1)} \otimes \zeta_{\bar{B}}^{+(2)}, \qquad \zeta_{\bar{A}}^{+(1)} \otimes \zeta_{\bar{B}}^{-(2)}, \tag{41}$$

where $\bar{A}, \bar{B} = 1, 2, \ldots, 8$. There are $64 + 64 + 64 + 64 = 256 = 2^{16/2}$ basis spinors of $\mathcal{C}\ell(16)$. Let us retain only 128 of them, say those of equation (40) that form a positive chiral representation (left handed) of so(16). The objects $\zeta_{\bar{A}}^{+(1)} \otimes \zeta_{\bar{B}}^{+(2)}$ and $\zeta_{\bar{A}}^{-(1)} \otimes \zeta_{\bar{B}}^{-(2)}$ span the 128-dimensional space S_{16}^+ of positive chiral spinors of $\mathcal{C}\ell(16)$.

As has been discussed in [13, 14], the space

$$\mathrm{so}\big(V_8^{(1)}\big) \oplus \mathrm{so}\big(V_8^{(2)}\big) \oplus V_8^{(1)} \otimes V_8^{(2)} \oplus S_8^{+(1)} \otimes S_8^{+(2)} \oplus S_8^{-(1)} \otimes S_8^{-(2)} = \mathrm{e}_8 \tag{42}$$

is the Lie algebra e_8 of E_8. Its dimension is exactly right, namely, $120 + 64 + 64 = 248$, and equation (42) is, in fact, a decomposition of

$$\mathrm{e}_8 = \mathrm{so}(16) \oplus S_{16}^+, \tag{43}$$

where S_{16}^+ forms the 128-dimensional irreducible representation of so(16). In equation (42), $V_8^{(1)}$ and $V_8^{(2)}$ are the vector representation spaces for the two copies of so(8), whereas $S_8^{+(1)}$, $S_8^{+(2)}$ and $S_8^{-(1)}$, $S_8^{-(2)}$ are the corresponding spinor representation spaces.

5. Discussion and conclusion

The exceptional group E_8 is somewhat enigmatic. Its smallest representation is the 248-dimensional adjoint representation, and the space in which E_8 acts as the isometry group cannot be determined without first defining E_8 [13]. Equations (42) and (43) demonstrate the curious fact that e_8 consists of the Lie algebra of rotations so(16) in a 16-dimensional vector space and the corresponding spinor space S_{16}^+ on which the generators of so(16) act. Both, the generators of so(16) and the spinors of S_{16}^+ are elements of $\mathcal{C}\ell(16)$, which contains e_8 as a subspace.

$\mathcal{C}\ell(16)$ is indeed a big space, probably too big to be useful for the unification of the currently known interactions and particles because it 'predicts' too many new particles and interactions. Therefore, it seems more reasonable to stay with (42), which also describes e_8 but with the corresponding spaces being subspaces of $\mathcal{C}\ell\big(V_8^{(1)}\big)$ or $\mathcal{C}\ell\big(V_8^{(2)}\big)$. The dimension of the latter is $2^8 = 256$, which is a modest in comparison with the dimensionality $2^{16} = 256 \times 256$ of $\mathcal{C}\ell(16)$. Although $\mathcal{C}\ell(16) = \mathcal{C}\ell(8) \otimes \mathcal{C}\ell(8)$, such a product is not necessary for the construction of e_8 according to equation (42). Only the direct products of the vector and spinor subspaces of the two copies of $\mathcal{C}\ell(8)$, but not of so(8), are necessary. We have arrived at $\mathcal{C}\ell(8)$ quite naturally by starting from the four-dimensional spacetime M_4. From its tangent vectors, we can generate the 16-dimensional Clifford algebra $\mathcal{C}\ell(1, 3)$, to which we can ascribe a physical interpretation as the space of oriented r-vectors associated with strings and branes [6–9, 15–17]. We consider $\mathcal{C}\ell(1,3)$ as a vector space $V_{8,8}$ in which one can perform 120 independent rotations. However, there are only 16 independent basis elements γ_A of $\mathcal{C}\ell(1,3)$, which, since they form an algebra, can generate only 32 independent rotations in $\mathcal{C}\ell(1,3)$, corresponding to left and right transformations in equation (9). The fact that the elements γ_A form an algebra is now an undesirable property because the product of two such gammas gives another gamma in the same set. Thus we have concentrated on the vector nature of $V_{8,8}$ and considered instead of $\{\gamma_A\}$ another basis $\{\Gamma_A\}$, whose objects do not close an algebra and can thus form an outer product $\Gamma_{AB} = \Gamma_A \wedge \Gamma_B$. Next, it has turned out to be convenient to split $V_{8,8}$ into two subspaces $V_8^{(1)}$ and $V_8^{(2)}$, with bases $\{\Gamma_{\bar{A}}^{(1)}\}$ and $\{\Gamma_{\bar{A}}^{(2)}\}$, $\bar{A} = 1, 2, \ldots, 8$. The bivectors $\Gamma_{\bar{A}\bar{B}}^{(1)}$ of $\mathcal{C}\ell(V_8^{(1)})$ and $\Gamma_{\bar{A}\bar{B}}^{(2)}$ of $\mathcal{C}\ell(V_8^{(2)})$ are generators of rotations in the corresponding vector spaces. We have also considered spinors of the Clifford algebras $\mathcal{C}\ell(V_8^{(1)})$ and $\mathcal{C}\ell(V_8^{(2)})$. Thus we have arrived at e_8 expressed in the form (42) by using only the elements of the two copies of $\mathcal{C}\ell(8)$ and the direct product of the vector and spinor representation spaces. Even if the decomposition (42) is a known result, it is fascinating to see how it fits into a straightforward extension of four-dimensional spacetime M_4 to the 16-dimensional space of r-volumes. The latter space, as shown in [6–9, 15–17], can be associated with strings and branes, fundamental physical objects that, in many respects, are very promising, although not the only ones considered in the literature, for unification and quantum gravity.

Acknowledgments

This work was supported by the Ministry of High Education, Science and Technology of Slovenia.

References

[1] See, e.g., Luciani J F 1978 *Nucl. Phys.* B **135** 111
Witten E 1981 *Nucl. Phys.* B **186** 412
Salam A 1982 *Ann. Phys., NY* **141** 352
Mecklenburg W 1984 *Fortschr. Phys.* **32** 207
[2] See, e.g., Green M B, Schwarz J H and Witten E 1987 *Superstring Theory* (Cambridge: Cambridge University Press)
Kaku M 1988 *Introduction to Superstrings* (New York: Springer)
Danielsson U 2001 *Rep. Prog. Phys.* **64** 51
Duff M J 2004 Benchmarks on the brane *Preprint* hep-th/0407175
Duff M J 1990 *Nucl. Phys.* B **335** 610
Duff M J and Lu J X 1990 *Nucl. Phys.* B **347** 394
[3] Lisi G 2007 Exceptionally simple theory of everything *Preprint* hep-th/0711.0770
[4] Gursey F 1978 *Proc. on Group Theoretical Methods in Physics* (Austin)
Baaklini N S 1980 *Phys. Lett.* B **91** 376
Bars I and Gunaydin M 1980 *Phys. Rev. Lett.* **45** 859
Castro C 2006 *Ann. Phys., NY* **321** 813
[5] Pezzaglia W M 1998 Polydimensional relativity, a classical generalization of the automorphism invariance principle *Clifford Algebras and their Applications in Mathematical Physics* ed V Dietrich (Dordrecht: Kluwer Academic) p 305–17 (*Preprint* gr-qc/9608052)
[6] Castro C *et al* 2000 *Chaos Solitons Fractals* **11** 1721 (*Preprint* hep-th/9912113)
[7] Pavšič M 2001 *Found. Phys.* **31** 1185 (*Preprint* hep-th/0011216)
[8] Pavšič M 2001 *The Landscape of Theoretical Physics: A Global View; From Point Particle to the Brane World and Beyond, in Search of Unifying Principle* (Dordrecht: Kluwer Academic)
[9] Pavšič M 2003 *Found. Phys.* **33** 1277 (*Preprint* gr-qc/0211085)
[10] Pavšič M 2005 *Phys. Lett.* B **614** 85 (*Preprint* hep-th/0412255)
[11] Pavšič M 2006 *Int. J. Mod. Phys.* A **21** 5905 (*Preprint* gr-qc/0507053)

[12] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory* (Cambridge: Cambridge University Press)
[13] Baez J 2002 The octonions *Bull. Am. Math. Soc.* **39** 142
[14] Castro C 2007 *Int. J. Geom. Meth. Mod. Phys.* **4** 1239
[15] Pavšič M 2007 *Found. Phys.* **37** 1197 (*Preprint* hep-th/0605126)
[16] Ansoldi S, Aurilia A, Castro C and Spallucci E 2001 *Phys. Rev.* D **64** 026003 (*Preprint* hep-th/0105027)
[17] Aurilia A, Ansoldi S and Spallucci E 2002 *Class. Quantum Grav.* **19** 3207 (*Preprint* hep-th/0205028)
[18] Teitler S 1965 *Suppl. Nuovo Cimento* **3** 1 and references therein
Teitler S 1965 *Suppl. Nuovo Cimento* **3** 15
Teitler S 1966 *J. Math. Phys.* **7** 1730
Teitler S 1966 *J. Math. Phys.* **7** 1739